Perturbational Variations in a Ballistic Missile or Satellite Orbit about an Oblate Earth

TOM T. KUMAGAI*
Hughes Aircraft Company, Fullerton, Calif.

This paper treats the problem of the oblateness effect on Earth satellites and ballistic missiles in terms of the expected deviations in position prediction relative to an ideal Keplerian orbit. The resulting closed form solutions in the variations in radius, in-plane angle, and lateral displacement have the following significant features: 1) easily instrumented in real-time tracking and prediction systems; 2) no restrictions on eccentricity, including orbits with escape velocities; and 3) offers a physical insight into the oblateness effect on freeflight trajectories. In addition, the second-order solutions are applicable for all angles of inclination with the possible exception of the region of critical inclination.

Nomenclature

V = oblate potential function
GM = gravitational product of Earth
r_e = mean value of the equatorial radius of Earth
r = instantaneous geocentric radius of the satellite
φ = colatitude angle of the satellite measured from Earth's rotating axis
J = second harmonic oblateness coefficient
δr = variation in the satellite radius caused by oblateness
θ = orbital in-plane angle of the satellite measured relative to the nodal line
β = nondimensional coefficient of angular momentum perturbation
ι = inclination angle of the orbital plane
ϵ = eccentricity of the orbit
l = $a(1 - \epsilon^2)$, length of the semilatus rectum
μ = angle of perigee of the orbit relative to the nodal line
ψ = lateral deflection angle of the satellite relative to the reference orbit plane
L = Lagrangian function
E = energy

Introduction

RECENTLY, a number of papers have been published in the technical literature concerning the oblateness effect on artificial Earth satellites. A partial list of the available references is given in the bibliography.[4,5,7,9-13,16,17] Baker and Makemson[1] and Sterne[15] devote several chapters in their respective books on the subject. With the possible exception of a few, the general approach falls into one of two categories: 1) the application of the variation of parameters technique on the classical orbital elements, or 2) the direct integration of the equations of motion, either by numerical integration or by transformation of the variables. The method of the variation of the parameters applied to the oblate Earth satellite problem yields results in terms of periodic and secular variations of the orbital elements. These results are useful if the primary interest in the oblate Earth satellite problem is a critical examination of the time dependence on the orbital elements. However, if the primary interest in the problem is to determine the oblateness effect on the predicted positions of a satellite or ballistic missile, the method of direct integration is more useful. Many, such as Brouwer,[2] Garfinkel,[3] and Vinti,[18] have performed direct integration on the equations of motion resulting in elliptic functions or elliptical integrals.

The analysis described in this paper also determines the solutions by the method of direct integration. Its purpose, however, is to determine simple solutions directly in terms of the satellite position and velocity parameters. These solutions are obtained by integrating the transformed force equations in terms of the difference values in the radius vector, in-plane angular displacement, and the crossrange displacement of the perturbed satellite orbit relative to a Keplerian elliptical orbit. The general solutions resulting from the analysis are also applicable for eccentricities of 1 and larger. Also, the solutions are understandable in terms of the geometry of the orbit. As a result, the solutions yield a physical insight into the oblateness effect on any orbiting satellite.

The equations of motion in terms of the difference values are obtained by using the truncated potential function that retains the second harmonic term in the latitude variation. Then the perturbations on the radius vector and the angular velocity are coupled together with the Lagrange equations of motion into a simple second-order differential equation. In addition, the application of the truncated potential function gives rise to a force normal to the orbit plane, provided that the satellite is neither polar nor equatorial. For the polar and equatorial orbits, the lateral displacement force equation is zero using the truncated potential function

$$V(r, \varphi) = -(GM/r_e)[(r_e/r) + J(r_e^3/r^3)(\tfrac{1}{3} - \cos^2\varphi)] \quad (1)$$

Oblateness Correction in the Radius Vector and the Angular Velocity

Since the Earth is flattened at the poles and bulges at the Equator, a latitude variation in mass creates a noncentral force field that can be represented by the potential function[6,8]

$$V(r,\varphi) = -(GM/r_e)[(r_e/r) - \tfrac{2}{3}J(r_e^3/r^3)P_2(\cos\varphi) - \tfrac{2}{5}H(r_e^4/r^4)P_3(\cos\varphi) + \tfrac{8}{30}K(r_e^5/r^5)P_4(\cos\varphi) + \ldots] \quad (2)$$

where

GM = gravitational product of Earth
r = distance to the center of Earth
r_e = mean equatorial radius of Earth
φ = geocentric colatitude of the satellite position
$P_2(\cos\varphi) = \frac{3}{2}\cos^2\varphi - \frac{1}{2}$
$P_3(\cos\varphi) = \frac{5}{2}\cos^3\varphi - \frac{3}{2}\cos\varphi$
$P_4(\cos\varphi) = \frac{1}{8}(35\cos^4\varphi - 30\cos^2\varphi + 3)$

and J, H, and K are the coefficients of the second, third, and fourth zonal harmonics.

The best present values of the geocentric constants based upon satellite data[6] are

$r_e = 6{,}378{,}145(1 \pm 11 \times 10^{-6})$ m
$(GM)^{1/2} = 1.996{,}501{,}5(1 \pm 11 \times 10^{-6}) \times 10^{-2}$ megm$^{3/2}$/sec
$J = (1623.41 \pm 4) \times 10^{-6}$
$H = (6.04 \pm 0.73) \times 10^{-6}$
$K = (6.37 \pm 0.23) \times 10^{-6}$

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* Senior Staff Engineer, Ground Systems Group. Member AIAA.

Since the uncertainty of the second zonal harmonic J is of the order of the H and K coefficients, the uncertainties in the results of the analysis using the truncated oblate potential expressed by Eq. (1) would tend to mask out the effects caused by oblateness using the H and K terms. Thus, the H and K terms will be omitted in this analysis.

Additional Assumptions

The analysis is performed in a drag-free atmosphere where Newtonian forces caused by the gravitational potential function, Eq. (1), are the only forces considered. Also, the gravitational attractions of the sun, moon, and nearby planets are ignored. Spitzer[14] has shown that the effect of the sun and moon alter the radial distance of a 500-mile altitude satellite approximately 1 and 2 ft, respectively, and alter the plane of rotation 0.05 and 1 deg/yr. These effects are negligible and are masked by the uncertainties in the geocentric oblate constants. In addition, the analysis is performed in a geocentric inertial frame of reference of the Earth.

Analysis

Let

r = Earth-centered radial distance of the satellite
r_u = associated unperturbed Earth-centered distance of the satellite which results from a pure inverse square law force field (Keplerian motion)
δr = $r - r_u$, which is the perturbation in r_u caused by oblateness
θ = in-plane angle of the satellite relative to equatorial crossing
β = nondimensional coefficient of perturbation
P = unperturbed angular momentum of the satellite associated with the reference orbit

Then in terms of the "orbital plane" the Lagrangian equations of motions are expressed as

$$\ddot{r} - r\dot{\theta}^2 + (GM/r_e)[(r_e/r^2) - 2J(r_e^3/r^4)P_2(\cos\varphi)] = 0 \quad (3)$$

$$(d/dt)(r^2\dot{\theta}) = -\tfrac{2}{3}JGM(r_e^2/r^3)(d/d\theta)P_2(\cos\varphi) \quad (4)$$

(The orbital plane is to be considered as the instantaneous orbital plane that contains the satellite and its velocity vector and is inclined at an instantaneous inclination angle ι relative to the equatorial plane. The lateral displacement effect caused by the Earth's oblateness exerts a torque upon the orbital plane causing a twisting motion of the instantaneous orbital plane. Thus, although the satellite motion is not planar, it can be treated as a planar orbit in which the orbit plane rotates about the Earth's axis.)

Eq. (4) can be integrated immediately in terms of the angular momentum of the unperturbed orbit:

$$r^2\dot{\theta} = P(1 + \beta) \quad (5)$$

where

$$P\beta = -\frac{2}{3}\,JGM\,r_e^2 \int_0^t \frac{1}{r^3}\,\frac{dP_2(\cos\varphi)}{d\theta}\,d\tau \quad (6)$$

Since this analysis is restricted to the order of the J harmonic coefficient in the oblateness effect, Eq. (6) can be integrated directly by using the solution of the Keplerian orbit.

$$\frac{1}{r} = \frac{1 + \epsilon\cos(\theta - \mu)}{a(1 - \epsilon^2)} \quad (7)$$

By transforming Eq. (6) from a time integration to a θ integration, Eq. (6) becomes

$$P\beta = -\frac{2}{3}\,JGM\,\frac{r_e^2}{P}\int_{\theta_0}^{\theta}\frac{1}{r}\,\frac{dP_2(\cos\varphi)}{d\theta}\,d\theta \quad (8)$$

where

$$d\tau = d\theta/\dot{\theta} = r^2 d\theta/P \quad (9)$$

The well-known relationship between φ and θ is

$$\cos\varphi = \sin\iota\,\sin\theta \quad (10)$$

Upon substitution of Eqs. (7) and (10) into Eq. (8), the solution for β can be found as

$$\beta = -\frac{2JGMr_e^2\sin^2\iota}{P^2a(1-\epsilon^2)}\left[\frac{\sin^2\theta}{2} - \epsilon\cos\mu\left(\frac{\cos^3\theta}{3} - \frac{1}{3}\right) + \epsilon\sin\mu\,\frac{\sin^3\theta}{3}\right] + C \quad (11)$$

where C is a constant of integration.

In the foregoing integration, the variations in $\sin\iota$, $\sin\mu$, and $\cos\mu$ were neglected since the solution is only of the order of the J harmonic coefficient.

Now reconsider Eq. (5). By considering the difference values between the actual geocentric radius r of the satellite and the associated unperturbed radius r_u determined by the Keplerian ellipse and also the corresponding difference values in θ, the use of Eq. (5) results in

$$\delta r = (r_u/2)[\beta - (\delta\dot{\theta}/\dot{\theta}_u)] \quad (12)$$

where

$$r = r_u + \delta r, \qquad \dot{\theta} = \dot{\theta}_u + \delta\dot{\theta} \quad (13)$$

The force equation of the difference values results from the substitution of Eq. (13) into the force equation, Eq. (3), as

$$\delta\ddot{r} - \dot{\theta}^2\delta r - 2r\dot{\theta}\delta\dot{\theta} - 2(GM/r^3)\delta r - 2JGM(r_e^2/r^4)P_2(\cos\varphi) = 0 \quad (14)$$

In Eq. (14) and all succeeding equations involving the difference values, the subscript u indicating the unperturbed values is dropped.

Eq. (14) can be simplified by transforming the first term into its equivalent θ derivative. This results in

$$\delta\ddot{r} = \dot{\theta}^2(d^2\delta r/d\theta^2) - 2(\dot{r}\dot{\theta}/r)(d\delta r/d\theta) \quad (15)$$

An expression for the second term in Eq. (15) can be found by considering the total energy equation. Using the second-order truncated potential function that results from Eq. (2), the total energy can be given as

$$E = (\dot{r}^2/2) + (r^2\dot{\theta}^2/2) - (GM/r_e)[(r_e/r) - \tfrac{2}{3}J(r_e^3/r^3)P_2(\cos\varphi)] \quad (16)$$

Upon substitution of Eq. (13) into Eq. (16),

$$\delta E = \dot{r}\delta\dot{r} + r^2\dot{\theta}\delta\dot{\theta} + r\dot{\theta}^2\delta r + (GM/r^2)\delta r + \tfrac{2}{3}JGM(r_e^2/r^3)P_2(\cos\varphi) \quad (17)$$

Eq. (17) represents the difference between the energy of the satellite in an oblate Earth orbit and the corresponding energy of the satellite in orbit about a spherical Earth.

Upon rearranging Eq. (17),

$$-\dot{r}\dot{\theta}(d\delta r/d\theta) = r^2\dot{\theta}\delta\dot{\theta} + r\dot{\theta}^2\delta r + (GM/r^2)\delta r + \tfrac{2}{3}JGM(r_e^2/r^3)P_2(\cos\varphi) - \delta E \quad (18)$$

Then substituting Eqs. (15) and (18) into Eq. (14), Eq. (14) simplifies into

$$(d^2\delta r/d\theta^2) + \delta r = \tfrac{2}{3}J(r_e^2/l)P_2(\cos\varphi) + 2(\delta E/P^2)r^3 \quad (19)$$

which becomes

$$(d^2\delta r/d\theta^2) + \delta r = \tfrac{2}{3}J(r_e^2/l)P_2(\cos\varphi) + 2\delta E(l^3/P^2) \times [1 - 3\epsilon\cos(\theta-\mu) + 6\epsilon^2\cos^2(\theta-\mu) - \ldots] \quad (20)$$

where $[1+\epsilon\cos(\theta-\mu)]^{-3}$ has been expanded in series form under the condition that the eccentricity is less than 1. This condition can be removed for orbits with eccentricities in the neighborhood of 1 and larger by choosing the reference orbit such that $\delta E = 0$. Eq. (20) has the solution

$$\delta r = A\sin\theta + B\cos\theta - \tfrac{1}{3}J(r_e^2/l)(\sin^2\iota\sin^2\theta - 2\sin^2\iota + 1) + \delta E(l^3/P^2)[2(1-4\epsilon^2) - 3\epsilon\theta\sin(\theta-\mu) - 4\epsilon^2\cos^2(\theta-\mu) + \ldots] \quad (21)$$

where A and B are constants of integration which are of the order of J. In the foregoing equations,

$$l = P^2/GM \quad (22)$$

which for elliptical orbits corresponds to the length of the semilatus rectum.

A word of caution must be noted on the initial condition restrictions. Since $\beta = 0$ at time $t = 0$, the initial conditions on δr and $\delta\dot\theta$ are related by the equation

$$2\delta r_0\dot\theta_0 = -r_0\delta\dot\theta_0 \quad (23)$$

In addition, Eq. (17) relates δE with the initial conditions on δr, $\delta\dot r$, and $\delta\dot\theta$. Thus, in general, either δr_0 or $\delta\dot\theta_0$ and either $\delta\dot r$ or δE can be specified to describe the reference Keplerian orbit from initial position and velocity data.

For most Earth satellite orbits where the eccentricity is sufficiently small compared to 1, a consistent set of initial conditions are

$$\begin{aligned}\delta r_0 &= \delta\dot r_0 = \delta\dot\theta_0 = 0\\ \delta E &= \tfrac{2}{3}JGM(r_e^2/r_0^3)P_2(\cos\varphi_0)\end{aligned} \quad (24)$$

This ideal set of initial conditions matches the Keplerian reference orbit to the perturbed orbit at time $t = 0$.

To determine the perturbation in the in-plane angle θ, the solution for δr, Eq. (21), is substituted into Eq. (12), which can be transformed into

$$d\delta\theta/d\theta = \beta - (2\delta r/r) \quad (25)$$

Upon integration, Eq. (25) has the solution

$$\begin{aligned}\delta\theta = \Big[J\frac{r_e^2}{l^2}\Big(\frac{2}{3} - \frac{3}{2}\sin^2\iota - \frac{2}{3}\epsilon\sin^2\iota\cos\mu\Big) - 4\delta E\frac{l^2}{P^2}(1-4\epsilon^2) - \frac{\epsilon}{l}(A\sin\mu + B\cos\mu) + C\Big]\theta +\\ \Big[2\frac{A}{l} - \frac{4}{3}J\frac{r_e^2}{l^2}\epsilon\sin^2\iota\sin\mu\Big]\cos\theta + \Big[-2\frac{B}{l} + \frac{2}{3}J\frac{r_e^2}{l^2}\epsilon\sin^2\iota\cos\mu\Big]\sin\theta +\\ \Big[\frac{1}{12}J\frac{r_e^2}{l^2}\sin^2\iota + \frac{1}{2}A\frac{\epsilon}{l}\sin\mu - \frac{1}{2}B\frac{\epsilon}{l}\cos\mu\Big]\sin 2\theta + \Big[2\delta E\frac{l^2}{P^2} - \frac{2}{3}J\frac{r_e^2}{l^2}(2\sin^2\iota - 1)\Big]\epsilon\sin(\theta-\mu) -\\ (A\cos\mu + B\sin\mu)\frac{\epsilon}{l}\sin^2\theta - 6\,\delta E\frac{l^2}{P^2}\epsilon\theta\cos(\theta-\mu) + \frac{11}{4}\delta E\frac{l^2}{P^2}\epsilon^2\sin 2(\theta-\mu) - \frac{3}{2}\delta E\frac{l^2}{P^2}\epsilon^2(\theta-\mu)\cos 2(\theta-\mu) -\\ 4\delta E\frac{l^2}{P^2}\epsilon^2(\theta-\mu) + D + 0(J\epsilon^3)\end{aligned} \quad (26)$$

where A and B are the constants of integration associated with δr, C is the constant of integration associated with β, and D is a new constant of integration.

Crossrange Oblateness Correction

From the physics of particles in a gravitational field, the gravitational force per unit mass is the negative of the potential gradient. Thus, since the chosen model for the oblate potential function is a function of latitude as well as the radius, the potential gradient has a component normal to the geocentric orbit plane. The normal component of force corresponding to the normal gradient gives rise to a lateral displacement of the satellite relative to the reference orbit plane. If the lateral displacement is attributed only to a differential change in the inclination angle, the instantaneous orbit plane eventually would become coplanar with the equatorial reference plane. However, in addition to the change in the inclination angle, the lateral displacement causes an angular change in the line of nodal crossing. Thus, the instantaneous orbit plane behaves analogous to a gyroscopic precession in the variational behavior of the inclination angle and the nodal line.

In addition to lateral potential gradient, the satellite experiences a centrifugal force. With respect to the unperturbed orbit plane, the centrifugal force has a component along the normal to the reference orbit plane. This component is of the order of the potential gradient in the crossrange direction of the reference plane.

The equation of motion in the cross direction could be determined from the foregoing discussion on physical behavior of particles in an oblate gravitational field; however, it is simpler to derive the equation of motion in the cross direction from the application of the complete Lagrangian function L, which is

$$L = \tfrac{1}{2}(\dot r^2 + r^2\dot\theta^2\cos^2\psi + r^2\dot\psi^2) - V(r,\varphi) \quad (27)$$

where, in addition to r, θ, φ, and V, which have been defined previously, ψ is the geocentric angle between the geocentric radius vector of the satellite and the reference orbit plane, measured in a plane perpendicular to the reference orbit plane. From Eq. (27), the lateral equation of motion is

$$(d/dt)[r^2(d\psi/dt)] + r^2\dot\theta^2\cos\psi\sin\psi = -(\partial V/\partial\psi) \quad (28)$$

By restricting ψ to small angles of the order of J and transforming the time derivatives to θ derivatives, Eq. (28) becomes

$$(d^2\psi/d\theta^2) + \psi = -(1/r^2\dot\theta^2)(\partial V/\partial\psi) \quad (29)$$

The right side of Eq. (29) can be written in a more convenient form by the transformation

$$\partial/\partial\psi = -(\cos\iota/\sin\varphi)(\partial/\partial\varphi) \quad (30)$$

where ψ is measured positive in the direction of the positive normal to the reference plane. The positive normal is defined as

$$\mathbf{n} = \mathbf{e}_r \times \mathbf{e}_\theta$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors associated with the radius vector and the local horizontal velocity of the satellite measured in the unperturbed orbit plane.

Using the truncated potential function and the relationship $\cos\varphi = \sin\iota\sin\theta$, the J order approximation of Eq. (29) is

$$(d^2\psi/d\theta^2) + \psi = -J(r_e^2/lr)\sin 2\iota\sin\theta \quad (31)$$

where $l = a(1-\epsilon^2)$. Upon substituting

$$\frac{1}{r} = \frac{1+\epsilon\cos(\theta-\mu)}{l}$$

Eq. (31) becomes

$$(d^2\psi/d\theta^2) + \psi = -J(r_e^2/l^2)\sin 2\iota[\sin\theta + \tfrac{1}{2}\epsilon\sin(2\theta-\mu) + \tfrac{1}{2}\epsilon\sin\mu] \quad (32)$$

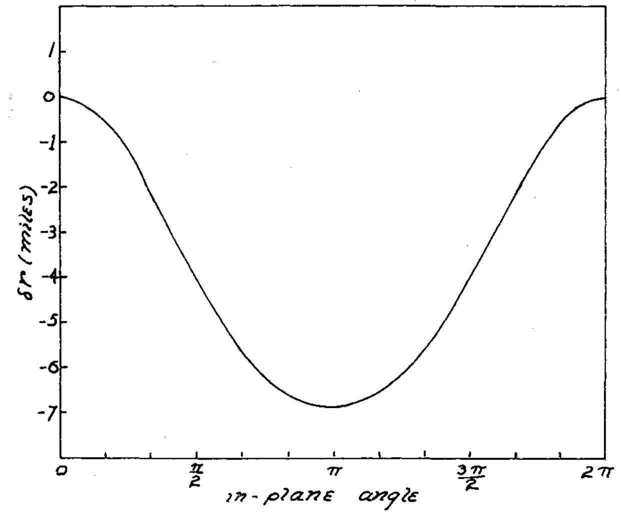

Fig. 1 Radial perturbation of a 100-mile altitude satellite ($\iota = 35°$, $\epsilon = 0$)

Eq. (32) has a general solution

$$\psi = \alpha_1 \sin\theta + \alpha_2 \cos\theta + \tfrac{1}{2} J(r_e^2/l^2) \sin 2\iota[\theta \cos\theta + (\epsilon/3) \sin(2\theta - \mu) - \epsilon \sin\mu] \quad (33)$$

Then the crossrange perturbation can be computed from Eq. (33) as $\delta R_c = r\psi$.

Discussion of Results

The problem of the oblateness effects on ballistic missiles or satellites has been analyzed from the viewpoint of differential errors relative to an ideal Keplerian orbit. The results in this form readily are applicable to real-time systems. For example, when analyzing the downrange errors of a ballistic missile, the oblateness variations can be considered as additive errors in addition to the errors determined by variations in the burnout parameters and errors caused by the drag effect.

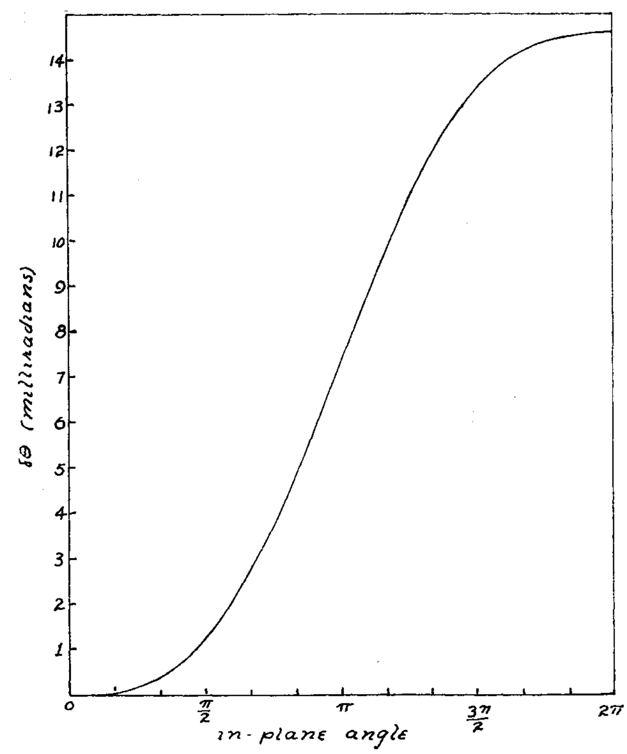

Fig. 2 In-plane angular perturbation of a 100-mile altitude satellite ($\iota = 35°$, $\epsilon = 0$)

In addition, by the proper choice of the difference energy constant and the initial conditions, the foregoing results can be applied to orbits with escape velocities.

The solutions are accurate to the order of the J harmonic coefficient and are not intended to determine the variations in the "fine structure" perturbations in the H and K harmonic coefficients. Also, the solutions are not applicable to long-period variations without correction on the reference orbit. For long-period variations without reference orbit correction, the secular and Poisson variations in the foregoing solutions eventually will exceed the basic assumption that the variations are of the order of the J harmonic coefficient. When the variations become larger than of J order, the difference equations used in the analysis require the addition of the quadratic and higher order terms in the variations.

Effect of Perturbation of the Inclination Angle

With respect to the inclination, a criticism has been received by the author of a possible error in logic in assuming

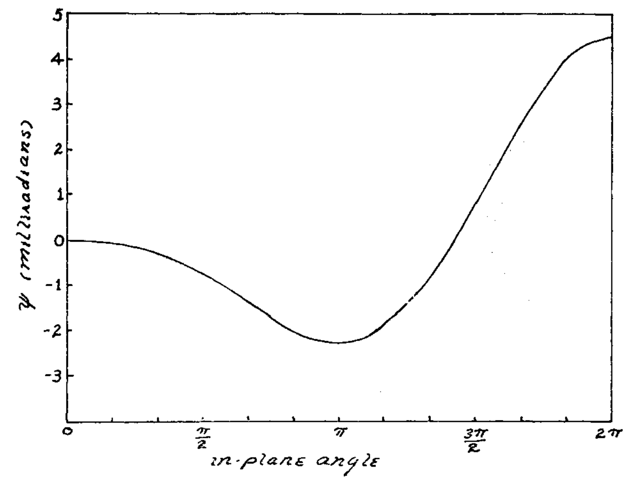

Fig. 3 Lateral angular perturbation of a 100-mile altitude satellite ($\iota = 35°$, $\epsilon = 0$)

that the variations in the inclination angle do not affect the integrations carried out in the analysis except to the order of J^2. This assumption has been used in the integration of Eqs. (8, 20, 25, and 32).

To determine the order of magnitude of the variation in the inclination angle $\delta\iota$, consider the constancy of the Z component of the perturbed angular momentum

$$r^2\dot{\theta} \cos\iota = \text{const} \quad (34)$$

or from Eq. (5)

$$P(1 + \beta) \cos\iota = \text{const} \quad (35)$$

Differentiating Eq. (35) with respect to θ results in

$$d\delta\iota/d\theta = \text{ctn}\,\iota(d\beta/d\theta) \quad (36)$$

Upon differentiating Eq. (8) with respect to θ, treating ι as a variable and substituting the result into Eq. (36) results in

$$d\delta\iota/d\theta = -\tfrac{1}{2}J(r_e^2/lr) \sin 2\iota \sin 2\theta + 0(J^2) \quad (37)$$

Thus, the variations in the inclination angle ι are only of the order of J, which justifies the original assumption that the inclusion of the variational behavior of the inclination angle gives rise to terms of the order of J^2 in the result.

Special Cases of Initial Conditions Chosen at Equatorial Crossing

To discuss the general behavior of the closed form solutions, it is of interest to consider the case where the reference

Keplerian orbit is matched with the perturbed orbit in position and velocity at the time of nodal crossing.

Then the solution in δr becomes

$$\delta r = Jl^2(r_e^2/r_0^3)[\tfrac{2}{3}\cos\theta - \tfrac{2}{3} + \epsilon\theta\sin(\theta-\mu) + \epsilon\sin\mu\sin\theta] + \tfrac{1}{3}J(r_e^2/l)[(1 - 2\sin^2\iota)(\cos\theta - 1) - \sin^2\iota\sin^2\theta] \quad (38)$$

From Eq. (38), δr has a maximum at $\theta = \pi$ and odd multiples of π for the orbits whose eccentricity is zero or for the orbits where the angle of perigee is $\pi/2$. For all other orbits with relatively small eccentricity, the maximum occurs in the neighborhood of $\theta = (2n-1)\pi$, where $n = 1, 2, 3 \ldots$. For the case $\epsilon = 0$,

$$\delta r_{\max} = (-2 + \tfrac{4}{3}\sin^2\iota)J(r_e^2/l) \quad (39)$$

The interesting feature about Eq. (39) is that $\delta r_{\max}$ is nonzero for the circular equatorial orbit. This is to be expected since there is a radial perturbation in r for circular equatorial orbits in accordance with Eq. (19) with $\iota = 0$.

Another interesting feature of Eq. (38) is that for orbits whose eccentricity is nonzero there is a Poisson variational term $[\theta\sin(\theta-\mu)]$, which can be attributed to the precession of the perihelion.

The variation in θ also includes a Poisson term for orbits with nonzero eccentricity. Other than the Poisson term, the physical behavior of the secular and periodic variations in θ can be examined by considering the simple case in which $\epsilon = 0$ and for which the initial conditions are chosen at the time of equatorial crossing. For this special case

$$\delta\theta = J(r_e^2/l)[(2 - \tfrac{3}{2}\sin^2\iota)\theta + \tfrac{1}{12}\sin^2\iota\sin 2\theta + 2(\tfrac{2}{3}\sin^2\iota - 1)\sin\theta] \quad (40)$$

The secular term added to the periodic terms gives rise to a positive displacement in the angular motion of the perturbed satellite relative to the motion of the satellite in the Keplerian orbit. In addition, the secular term in Eq. (40) is the major contributor to the precession of the perihelion for orbits with nonzero eccentricity.

For the lateral perturbation, if the initial conditions are chosen at the time of equatorial crossing with the perturbed position and velocity matched with the Keplerian orbit, the equation for the lateral perturbation becomes

$$\psi = \tfrac{1}{2}J(r_e^2/l^2)\sin 2\iota[\theta\cos\theta - (1 + \tfrac{2}{3}\epsilon\cos\mu)\sin\theta + \tfrac{4}{3}\epsilon\sin\mu\cos\theta + (\epsilon/3)\sin(2\theta-\mu) - \epsilon\sin\mu] \quad (41)$$

The most significant term in the lateral perturbation is the Poisson term $(\theta\cos\theta)$, which is the perturbation that gives rise to the secular variation in the instantaneous nodal line. To the J order of approximation, the lateral perturbation is the largest contributor to the variation in the node and the variation in the inclination angle. Since ψ is measured positive in the direction of the positive normal, the nodal variation results in the regression of the node. The regression of the node and the precession of the perihelion can be studied from the results of this analysis by the use of vector analysis to transform the variations in position into variations in the node and the perihelion.

Figures 1–3 are plots of the variations given in Eqs. (38, 40, and 41), respectively, for the case $\epsilon = 0$.

To obtain a physical insight in the oblateness effect on orbiting satellites, consider Eqs. (38, 40, and 41) and the corresponding three figures simultaneously for the special case $\epsilon = 0$. Initially, let two particles coincide in position and velocity at the point of equatorial crossing moving in a northerly direction at the inclination angle ι. Let one of the particles move in a Keplerian force field in a circular orbit. Let the other particle move in the oblate force field. Then, during the first half of the orbit, the particle in the oblate force field decreases its geocentric radius and moves ahead of the particle in the circular orbit. As the particle in the circular orbit crosses the Equator moving south, the particle in the oblate field has minimum radius. Then, during the next half cycle, the radius of the particle in the oblate field increases until at the end of this half cycle the radius matches the initial radius. The angular displacement between the two particles monotonically increases with a greater increase in the second half of the orbit relative to the first half.

In addition, according to Eq. (37), the instantaneous inclination angle of the perturbed orbit oscillates with a frequency that is twice the frequency of the orbit cycle. During each odd quarter cycle while the inclination angle is decreasing, the nodal line is regressing at a rate related to the decrease in the inclination angle. Then, during each even quarter cycle while the inclination angle is increasing, the rate of regression of the node, although monotonic, diminishes. This physical picture is consistent with the solution in the lateral displacement angle when transformed into nodal variations. Thus, the motion behaves analogous to a gyroscopic precession.

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